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Bound states in bottomless potentials

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We consider classical and quantum dynamics on potentials that are asymptotically unbounded from below. By explicit construction we find that quantum bound states can exist in certain bottomless potentials. The classical dynamics in these potentials is novel. Only a set of zero measure of classical trajectories can escape to infinity. All other trajectories get trapped as they get further out into the asymptotic region.

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Certain potentials that do not have any local minima and are unbounded from below in the asymptotic region can still exhibit stable classical orbits. Specifically, consider a particle in two spatial dimensions with position $\mathbf{x}(t) = (x(t), y(t))$ and Lagrangian

$$L = \frac{1}{2} m \mathbf{v}^2 - V(\mathbf{x}) \tag{1}$$

where

$$V(\mathbf{x}) = \frac{k_x}{2} x^2 - \frac{k_y}{2} y^2 + \frac{c}{2} x^2 y^2 \tag{2}$$

with k_x, k_y and c being positive parameters. The only extremum of the potential is at $\mathbf{x} = 0$ and this is a saddle point. The positive eigenvalue at the saddle point is along the x direction and the negative eigenvalue is along the y direction. The potential goes to $-\infty$ for $x^2 < k_y/c$ and $y^2 \to \infty$. Hence the potential is "bottomless". Yet it is not hard to show that there exist classical orbits that are bounded and stable. These solutions are of the form:

$$x(t) = A \cos(\omega t) \ , \quad y(t) = 0 \tag{3}$$

where $\omega^2 = k_x/m$ and A lies in specific bands determined by solutions of the Mathieu equation [1].

The existence of classically bounded and stable orbits on a bottomless potential motivates us to consider the possibility of quantum bound states on a bottomless potential. Several potentials that are unbounded from below have already been considered in the quantum mechanics literature. These include the famous example of the Coulomb potential. Note, however, that the Coulomb potential is bottomless because there is a singular point where the potential goes to $-\infty$. In contrast, the potentials we will consider (*eg.* in eq. (2)) go to $-\infty$ in the asymptotic region and are otherwise non-singular. Furthermore, we know that the $l = 0$ wavefunctions of the hydrogen atom have discontinuous first derivative at the origin, while the wavefunctions with non-zero angular momentum vanish at the origin due to the centrifugal barrier. In our case, the wavefunction will be analytical throughout and its existence cannot be attributed to a centrifugal barrier.

The idea of the construction is quite simple. We write the two dimensional Schrodinger equation in the form:

$$V(\mathbf{x}) = E - \frac{\hbar^2}{2m} [\nabla^2 F - (\nabla F)^2] \tag{4}$$

in standard notation and with $F(\mathbf{x})$ defined via

$$\psi = e^{-F} \tag{5}$$

where $\psi(\mathbf{x})$ is the wavefunction.

We would like a solution to eq. (4) for a bottomless potential and a discretely normalizable wavefunction. To accomplish this, we choose

$$F = c + \alpha x^2 + \beta y^2 + \gamma x^2 y^2 \tag{6}$$

where α, β, and γ are real positive numbers to ensure that ψ is normalizable, and $\exp(-c)$ is the normalization constant. Inserting this choice of F in eq. (4) gives:

$$\begin{aligned} V(\mathbf{x}) \ &= \ \epsilon + \frac{\hbar^2}{m} \Big[(2\alpha^2 - \gamma) x^2 + (2\beta^2 - \gamma) y^2 \\ &+ \ 4(\alpha + \beta)\gamma x^2 y^2 + 2\gamma^2 (x^2 + y^2) x^2 y^2 \Big] \end{aligned} \tag{7}$$

where

$$\epsilon \equiv E - \frac{\hbar^2}{m} (\alpha + \beta) \ . \tag{8}$$

If we fix the zero of the potential to be at the origin, this gives the energy eigenvalue of the state. Now if we choose $\gamma > 2\alpha^2$ and/or $\gamma > 2\beta^2$, the potential is bottomless along the x axis and/or y axis respectively. This shows an explicit example of a bottomless potential in which there is at least one bound state. There are many other possibilities that can easily be constructed in a similar way.

There are some features of the solution that are worth pointing out. It is easily seen that the state is not an angular momentum ($L_z = -i\hbar(x\partial_y - y\partial_x)$) eigenstate. However the expectation value of the angular momentum operator vanishes. A peculiar feature of the potential is that it depends on the mass of the particle since the mass

m cannot be absorbed by rescaling the parameters α, β and γ. An interesting feature of the wavefunction is that it has the same sign everywhere *i.e.* it has no nodes. Therefore it must be the ground state wavefunction.

One intuitive way to understand the existence of a bound state is to examine the potential along one of the "escape" directions. For example, consider the potential in eq. (7) along the y axis. In this direction ($x = 0$), the potential is falling off and getting deeper in proportion to $-y^2$. However, the width along the x direction is also decreasing in proportion to $1/\sqrt{y^4} \propto y^{-2}$. Hence, if we consider the x dependence of the wavefunction for large values of y, it corresponds to a harmonic oscillator with angular frequency proportional to y^2. Therefore the energy "cost" due to the squeezing in the x direction grows as y^2 and can be larger than the energy "gain" due to rolling in the y direction. This argument is basically saying that if there is a hole in a two dimensional potential, quantum particles may not be able to escape through the hole if it is sufficiently narrow.

The quantum behaviour is in contrast to the classical particle which can always escape by rolling along the y axis. However, if a particle starts rolling in the y direction but with $x \neq 0$, it will oscillate in the x direction as it is rolling in the y direction. For large values of y and small values of x, the equations of motion are:

$$\ddot{x} \simeq -\nu^2 y^4 x \ , \quad \ddot{y} \simeq -2\nu^2(x^2y^2 - \delta^2)y \tag{9}$$

where

$$\nu \equiv \frac{2\hbar}{m} \ , \quad \delta \equiv \frac{\sqrt{\gamma - 2\beta^2}}{2\gamma} \ . \tag{10}$$

Hence, the sign of the force in the y direction tends to drive the particle toward the asymptotic region only while the particle lies in the region $xy < \delta$. While the particle lies outside the hyperbola (*i.e.* when $xy > \delta$), it experiences a restoring force which tends to bring the particle closer to the origin. If the particle is initially inside the hyperbola, it rolls down to larger and larger values of y, and eventually the particle orbit will increasingly lie outside the hyperbola. Then the particle will perform oscillations in both the x and y directions at some large value of y. This argument can be formalized by writing:

$$y(t) = Y + f(t) \tag{11}$$

where $Y > 0$ is a large constant value. We assume that $f(t) \ll Y$ and that $x(t)$ remains small. Then we can do a linearized analysis in $f(t)$ and obtain:

$$\ddot{x} \simeq -\nu^2 Y^4 x \ , \quad \ddot{f} \simeq -2\nu^2(x^2Y^2 - \delta^2)Y \tag{12}$$

Then $x(t) = A\cos(\nu Y^2 t)$ and the $f(t)$ equation can also be easily integrated. Periodic solutions for $f(t)$ will be obtained when the average of the right-hand side of the f equation vanishes. This happens when $A^2Y^2 = 2\delta^2$ and the solution is:

$$f(t) = \frac{\delta^2}{2Y^3}(\cos(2\nu Y t) - 1) \tag{13}$$

where we have also imposed the initial conditions $f(0) = 0$, $\dot{f}(0) = 0$. Hence periodic solutions do appear to leading order in the linearized approximation. A particle that starts rolling but is off-axis will eventually get caught in a periodic or quasi-periodic orbit and will not escape.

If the above arguments are correct, it implies two interesting corollaries. The first is that the potential in eq. (7) does not have any continuum states since the arguments apply to all states and no state would be able to escape to infinity. In principle, one could modify the potential by cutting it off at some large value – that is, by setting $V(\mathbf{x}) = V_{max}$ in regions where the original potential (eq. (7)) exceeds V_{max}. The new potential would be bounded from above and continuum states would then exist. The second corollary of the above argument is that the potential in eq. (2) does not have any bound state solutions. This is because the energy gain along the escape path is still proportional to y^2 but the energy cost due to the squeezing only grows like $\sqrt{y^2} = y$.

If we include dissipative forces, then all classical particles will escape to infinity since this is the lowest point on the potential. Dissipative effects in the quantum problem can, however, only bring the particle into its ground state. This is just as in the Coulomb case where classically the atom can collapse due to emission of electromagnetic radiation but quantum mechanically it can only settle into its ground state.

Generally speaking, bottomless potentials in the field theory context are thought to be sick. Our construction here raises the possibility that some field theories with bottomless potentials may nonetheless have reasonable interpretations.

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[1] M. Salem and T. Vachaspati, hep-th/0203037 (2002).